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LETTER TO THE EDITOR

# A $q$-analogue of the Campbell–Baker–Hausdorff expansion

Jacob Katriel† and Allan I Solomon
Faculty of Mathematics, The Open University, Milton Keynes, MK7 6AA, UK

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**Abstract.** We describe a $q$-analogue of the Campbell-Baker-Hausdorff formula and give an explicit expansion to fourth order.

The CBH formula [1–4], gives an expansion for the operator $z$ in terms of $x$ and $y$, where

$$\exp(x)\exp(y) = \exp(z). \tag{1}$$

The operator $z$ is given as an infinite series of commutators of $x$ and $y$ so that when $x$ and $y$ are members of a Lie algebra the exponent is expressible in an algebraically closed form. In this latter case we may write

$$\exp(tx)\exp(ty) = \exp(C(tx,ty)) \tag{2}$$

where

$$C(tx,ty) = \sum_{n=1}^{\infty} c_n(x,y)t^n. \tag{3}$$

The expansion to fifth order has recently been given by Czyz [5]:

$$c_1(x,y) = x + y \tag{4}$$

$$c_2(x,y) = \tfrac{1}{2}[x,y] \tag{5}$$

$$c_3(x,y) = \tfrac{1}{12}[x,[x,y]] - \tfrac{1}{12}[y[x,y]] \tag{6}$$

$$c_4(x,y) = -\tfrac{1}{24}[x,[y,[x,y]]] \tag{7}$$

$$\begin{aligned} c_5(x,y) = {}& -\tfrac{1}{720}[x,[x,[x,[x,y]]]] - \tfrac{1}{120}[x,[x,[y,[x,y]]]] + \tfrac{1}{360}[y,[x,[x,[x,y]]]] \\ & - \tfrac{1}{360}[x,[y,[y,[x,y]]]] + \tfrac{1}{120}[y,[x,[y,[x,y]]]] + \tfrac{1}{720}[y,[y,[y,[x,y]]]]. \end{aligned} \tag{8}$$

Deformations of groups and the corresponding algebras (Hopf algebras) have recently been the object of much activity in theoretical physics (see, for example, [6]). Applications of these so-called quantum groups range from the exact solution of problems in

† Permanent address: Department of Chemistry, Technion–Israel Institute of Technology, Haifa 32000, Israel.

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statistical mechanics [7] to the description of new exotic 'squeezed states' in quantum optics [8]. These applications involve the introduction of quantum operators whose commutation relations differ from those of conventional bosons, and analogues of the exponential function whose properties are less familiar than those of the conventional one. Nevertheless, such $q$-analogues of classical analysis have been well studied in the mathematical literature [9–11]. In this letter we show how a natural analogue of the commutator and of the exponential function satisfy a $q$-version of the CBH result.

In the $q$-analogue of analysis, the $q$-differentiation operator ${}_qD_x$ is defined by

$$ {}_qD_x f(x) = (f(qx) - f(x))/(q-1)x. \tag{9}$$

(It is useful to retain both the $q$ and $x$ subscripts in the definition.)

The analogue of the exponential function $E_q(x)$ then satisfies

$$ {}_qD_x E_q(\alpha x) = \alpha E_q(\alpha x) \tag{10}$$

whence

$$ E_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!} $$

where $[n]! = [n].[n-1]\ldots[1]$ and $[n] = (q^n - 1)/(q-1)$. (Note the difference in definition here from that current among physicists, where the prevalent usage is $[n]_q = (q^n - q^{-n})/(q - q^{-1})$, with a corresponding $\exp_q(x)$ satisfying

$$ {}_qD_x^{\mathrm{P}} \exp_q(\alpha x) = \alpha \exp_q(\alpha x) \tag{11}$$

with ${}_qD_x^{\mathrm{P}} f(x) = (f(qx) - f(x/q))/(x(q - 1/q)$. This latter definition has the advantage of being symmetrical under the interchange $q \leftrightarrow 1/q$ but lacks some of the elegant properties of the simpler, classical definition.)

Although the $q$-exponential function $E_q(x)$ does not possess the usual multiplicative property of the conventional exponential in general, Cigler [12] has shown that when the '$q$-mutator' $[y,x]_q := yx - qxy$ vanishes, then indeed

$$ E_q(x)E_q(y) = E_q(x+y). \tag{12}$$

This result suggests that a $q$-analogue of the CBH formula exists, namely;

$$ E_q(x)E_q(y) = E_q(z) \tag{13}$$

where $z$ depends only on the $q$-mutator $[y,x]_q$.

We evaluate explicitly the first five coefficients $c_{ij}$ corresponding to the coefficients $c_n$ in (3) above. Define

$$ \begin{aligned} E_q(\alpha x)E_q(\beta y) = E_q(&\alpha x + \beta y + \alpha\beta c_{11} + \alpha^2\beta c_{21} + \alpha\beta^2 c_{12} \\ &+ \alpha^3\beta c_{31} + \alpha^2\beta^2 c_{22} + \alpha\beta^3 c_{13} + \ldots). \end{aligned} \tag{14}$$

Direct comparison of the coefficients gives the following recursive system:

$$xy = c_{11} + \{xy\} \tag{15}$$

$$\frac{1}{[2]!}x^2y = c_{21} + \{xc_{11}\} + \{x^2y\} \tag{16}$$

$$\frac{1}{[2]!}xy^2 = c_{12} + \{yc_{11}\} + \{xy^2\} \tag{17}$$

$$\frac{1}{[3]!}xy^3 = c_{13} + \{yc_{12}\} + \{y^2c_{11}\} + \{xy^3\} \tag{18}$$

$$\frac{1}{[3]!}x^3y = c_{31} + \{xc_{21}\} + \{x^2c_{11}\} + \{x^3y\} \tag{19}$$

$$\frac{1}{([2]!)^2}x^2y^2 = c_{22} + \frac{c_{11}^2}{[2]} + \{xc_{12}\} + \{yc_{21}\} + \{xyc_{11}\} + \{x^2y^2\} \tag{20}$$

$$\dots$$

where $\{abc\dots\}$ is a modified monomial symmetric function on $n$ symbols $a, b, c, \dots$; that is, the sum of all *different* permutations divided by $[n]!$. Thus,

$$\{xc_{11}\} = (xc_{11} + c_{11}x)/[2]! \tag{21}$$

$$\{xy^3\}(= \{xyyy\}) = (xy^3 + yxy^2 + y^2xy + y^3x)/[4]!. \tag{22}$$

We find for the first four terms

$$c_{11} = \frac{-1}{(q+1)}[y, x]_q \tag{23}$$

$$c_{21} = \frac{-q}{[3]!(q+1)}[[x, y]_q, x]_q \tag{24}$$

$$c_{12} = \frac{-q}{[3]!(q+1)}[y, [x, y]_q]_q \tag{25}$$

$$c_{13} = \frac{-q^2}{[4]!(q+1)}([y, [y, [x, y]_q]_q]_q + [[[y, x]_q, y]_q, y]_q) \tag{26}$$

$$c_{31} = \frac{-q^2}{[4]!(q+1)}([x, [x, [y, x]_q]_q]_q + [[[x, y]_q, x]_q, x]_q) \tag{27}$$

$$c_{22} = \frac{1}{[4]!(q+1)}\left\{\left(\frac{-q^2}{2(q+1)^2}\right)(q(t_1 + t_6) + (q^2 + q + 1)(t_3 + t_4)) + q(-qt_2 + (q^2 - q + 1)t_5)\right\} \tag{28}$$

where

$$t_1 = \tfrac{1}{2}\{[x, [[x, y]_q, y]_q]_q + [[x, [x, y]_q]_q, y]_q\} \tag{29}$$

$$t_2 = \tfrac{1}{2}\{[[[x, y]_q x]_q, y]_q + [x, [y, [x, y]_q]_q]_q\} \tag{30}$$

$$t_3 = \tfrac{1}{2}\{[[[x, y]_q y]_q, x]_q + [x, [y, [y, x]_q]_q]_q\} \tag{31}$$

$$t_4 = \tfrac{1}{2}\{[[[y, x]_q x]_q, y]_q + [y, [x, [x, y]_q]_q]_q\} \tag{32}$$

$$t_5 = \tfrac{1}{2}\{[[[y, x]_q y]_q, x]_q + [y, [x, [y, x]_q]_q]_q\} \tag{33}$$

$$t_6 = \tfrac{1}{2}\{[y, [[y, x]_q, x]_q]_q + [[y, [y, x]_q]_q, x]_q\}. \tag{34}$$

It is straightforward to verify that in the limit $q \to 1$ these expressions agree with the conventional CBH terms given in [5]. Further, due to various identities among the $q$-mutators the expressions depend only on [y,x] There is no analogue of the $c_{13}$ and $c_{31}$ terms—they vanish in the limit. Agreement of the $c_{22}$ term is achieved by use of the limiting values

$$t_1 + t_6 = t_2 + t_5 = t_3 + t_4 = 0. \tag{35}$$

It may be proved that the coefficients in the expansion (14) satisfy the following invariance property:

$$c_{ij}(y, x, 1/q) = (-1)^{i+j+1} c_{ji}(x, y, q). \tag{36}$$

It is of interest to note that no naive analogue of the CBH result exists for the more usual form of $q$-exponential function defined in (11) above. This follows since, in the corresponding expansion for $\exp_q$,

$$\begin{aligned} \exp_q(\alpha x)\exp_q(\beta y) = \exp_q(&\alpha x + \beta y + \alpha\beta c_{11} + \alpha^2\beta c_{21} + \alpha\beta^2 c_{12} \\ &+ \alpha^3\beta c_{31} + \alpha^2\beta^2 c_{22} + \alpha\beta^3 c_{13} + \ldots) \end{aligned} \tag{37}$$

the vanishing of the first term $c_{11}$ requires the modified $Q$-mutator

$$[y, x]_Q = yx - Qxy$$

to vanish, where $Q = q + 1/q - 1$. This is not consistent with the vanishing of the higher-order $c_{ij}$ terms.

Finally, we note that the $q$-CBH expansion presented here sheds new light on the form of the conventional CBH formula, since much of the structure in the latter expansion is lost due to the (anti-)symmetry of the conventional commutator, absent here.

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